On Equality of Lines with Positional Uncertainty

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1. Introduction

Geometric reasoning in vector based geographic information systems (GIS) is based on Euclidean geometry. Euclid's first postulate, saying that *the line determined by two points is unique*, makes geometric constructions unambiguous, and, e.g., allows for specifying line features and polygons by tupels of points. It relies on the assumption that the location of a point can be unambiguously and accurately described by a coordinate pair. In contrast to this, the geographic location of a point in space can not be exactly determined, but is subject to positional uncertainty (Goodchild, 2000). One way to establish a uniqueness property for points and lines with positional uncertainty in analogy to Euclid's first postulate is to introduce fuzzy equality predicates measuring a *degree of equality* of points and lines with positional uncertainty, respectively (Wilke 2009). This abstract focuses on equality of *lines* with uncertainty in location, and derives a list of four requirements to such an equality predicate. We focus on the simplest form of positional uncertainty, namely on location constraints: In this case, the true location of a point or a line is delineated by one or several regions in a coordinate space, and no distribution or weighting of the coordinate points within that regions is given. For many practical cases, assuming a single simply connected and regular constraining region is sufficient. We define *extended points (extended lines)* to be points (lines) whose exact location is unknown, and whose set of possible locations is specified by a simply connected regular subset of a coordinate space (parameter space). We interpret the incidence relation (on-relation) between extended points and extended lines by the subset relation of the underlying coordinate space.

2. Related Work

The topic of consistent geometric reasoning with points and lines that have positional uncertainty is a longstanding problem in vector based GIS. Some of the oldest references go back to Perkal (1966), Pullar (1991), or Chrisman, Dougenik and White (1992), introducing the concept of epsilon tolerance and fuzzy tolerance into geometric reasoning. Most of the more recent contributions, e.g. Leung, Ma and Goodchild (2004) or Heuel (2004), focus on positional uncertainty that stems from stochastic variability. Here, it is common to assume a Gaussian probability density function with unique mean to describe the uncertainty in position. In contrast to these approaches, the present work aims at integrating the concept of positional tolerance into the axiomatic fundament of geometry. As a first step this abstract addresses the formalization of the most fundamental axiom of all classical geometries, namely Euclid's first postulate, for extended points and lines. Adopting an axiomatic approach has the advantage that consistency issues can be investigated with the tools of mathematical logic.

3. Requirements to an equality relation of extended lines

In the following subchapters 3.1 - 3.6 we derive a list of four requirements to an equality relation between extended lines that has to be taken into account when formalizing Euclid's first postulate for extended primitives.

3.1 Equality of extended lines is graduated

Figure 1 illustrates that Euclid's first postulate "*The line determined by two points is unique*" does not apply for extended points and extended lines: If two distinct extended points P, Q lie on the extended lines L_1 and L_2, L_1 and L_2 are not necessarily equal.
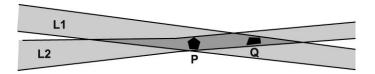

Figure 1. Two extended points do not uniquely determine an extended line.

Yet, L_1 and L_2 are in general "closer together", i.e. "more equal", than arbitrary extended lines that have only one or no extended point in common. This fact can be modeled by an equality relation that allows not only Boolean values, but values in the interval $[0,1]$.

The standard way to model graduated equality relations, or, more generally, equivalence relations, is to use *fuzzy equivalence relations* (Hajek 1998). A fuzzy equivalence relation is a fuzzy relation $eq : D^2 \rightarrow [0,1]$ on a domain D that is reflexive, symmetric and transitive:

$$eq(x,x) = 1 \tag{1}$$

$$eq(x,y) = eq(y,x) \tag{2}$$

$$eq(x,y) * eq(y,z) \leq eq(x,z), \tag{3}$$

where $*$ denotes a fuzzy logical AND operator, called *T-norm,* and $\leq$ replaces the implication operator[1]. An example of a prominent T-norm is the Łukasiewicz T-norm, which we denote by $\otimes$. It is defined by

$$\otimes : [0,1]^2 \rightarrow [0,1], \quad x \otimes y = \max\{x + y - 1, 0\}. \tag{4}$$

3.2 Equality of extended lines is not transitive

DeCoeck and Kerre (DeCoeck and Kerre 2003) pointed out that some spatial equality relations are not transitive, and thus cannot be modeled by fuzzy equivalence relations. A typical example of a non-transitive spatial equality relation is generated by measurement with limited accuracy: Two points p and q of the underlying coordinate space are indiscernible, if their distance lies below a threshold c. The relation $\overset{c}{\approx}$ of indiscernibility is not transitive (Figure 2).

$$p \overset{c}{\approx} q, \quad q \overset{c}{\approx} r, \quad p \overset{c}{\not\approx} r. \tag{5}$$

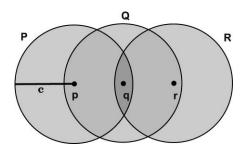[1] $(a \rightarrow b) = 1$ iff $a \leq b$ (Hajek 1998)

Figure 2. *p* and *q* are indiscernible, *q* and *r* are indiscernible, but *p* and *r* are discernible.

This phenomenon is commonly referred to as Poincaré Paradox (Gerla 2008). As can be seen in Figure 2, the relation $\overset{c}{\approx}$ associates with each coordinate point *p* a region of indiscernibility *P*, which can be interpreted as a location constraint to the unknown true location of the coordinate point *p*. This example shows that location constraints are susceptible to the Poincaré Paradox and that fuzzy equivalence relations are not suitable for modeling graduated equality in this context.

Instead, *tolerance relations* are good candidates: A (crisp or fuzzy) tolerance relation $\approx$ is a (crisp or fuzzy) relation that is reflexive and symmetric, but not necessarily transitive. It associates with each exact coordinate point *p* a set *P* of points that are not discernible from *p* w.r.t. $\approx$. If the set *P* is simply connected and regular, it can be interpreted as an extended point, and $\approx$ can be interpreted as a graduated equality relation between extended points. Due to the missing transitivity property regions of indiscernibility do not form disjoint equivalence classes, but may overlap. This is consistent with the fact that regions defined by location constraints may overlap.

3.3 Equality of extended lines is not non-transitive

As pointed out in the last subchapter, the transitivity property of fuzzy equivalence relations is too restrictive for our purpose. In the following we will see that tolerance relations are too broad for our purpose: they do not provide enough information to relate the two lines L_1 and L_2 of Figure 1 by a graduated equality predicate.

Since we defined extended points *P* and *Q* as location constraints to exact coordinate points *p* and *q*, respectively, the pair (P,Q) can be interpreted as a location constraint to the unique exact line determined by *p* and *q*. In other words, the extended lines L_1 and L_2 are related via the extended line generated by (P,Q). Looking at the axioms (1)-(3) of fuzzy equivalence relations, we see that the axiom of transitivity (3) is the only tool for relating two objects via a third. Dropping it deprives us of any means to derive a graduated equality value for L_1 and L_2.

So far, we found three requirements for a graduated equality relation between extended lines: It should be 1. graduated, 2. not transitive, and 3. not non-transitive. In a Boolean calculus, requirements 2 and 3 would produce a contradiction. In a fuzzy logical calculus, this issue can be resolved by graduating the transitivity property.

3.4 Equality of extended lines is weakly transitive

G. Gerla (2008) shows that for modeling the Poincaré paradox in a graduated context, transitivity need not be dropped completely, but may instead be replaced with a weaker form:

$$eq(x, y) * eq(y, z) * dis(y) \leq eq(x, z). \tag{6}$$

Here $dis(y) \in [0,1]$ is a lower-bound measure for the degree of transitivity that is permitted by y. We call a pair

$$\left(eq : D^2 \to [0,1], \quad dis : D \to [0,1] \right) \tag{7}$$

satisfying reflexivity (1), symmetry (2) and weak transitivity (6) an *approximate fuzzy $*$-equivalence relation*[2].

3.5 Equality of extended lines is inverse to their distance

In a spatial context, it is reasonable to require a graduated equality relation (often called similarity measure) to be inverse to a distance measure. Gerla (2008) shows that this requirement can be maintained for *approximate fuzzy $*$-equivalence relations* in a slightly adapted form: If the Łukasiewicz T-norm $\otimes$ is chosen for $*$, eq and dis can be represented by

$$eq(x, y) = \max \{ 1 - \delta(x, y), 0 \}, \tag{8}$$

$$dis(y) = \max \{ 1 - s(y), 0 \}. \tag{9}$$

Here $s : D \to \square^+$ is a size measure, and $\delta : D^2 \to \square^+$ is a distance measure for extended sets: The pair (δ, s) is called a *pointless pseudo-metric*, and satisfies the following axioms:

$$\delta(x, x) = 1, \tag{10}$$

$$\delta(x, y) = \delta(y, x), \tag{11}$$

$$\delta(x, y) + \delta(y, z) + s(y) \geq \delta(x, z). \tag{12}$$

The inequality (12) is a weak form of the triangle inequality. It corresponds to the weak transitivity

$$eq(x, y) \otimes eq(y, z) \otimes dis(y) \leq eq(x, z) \tag{13}$$

of the approximate fuzzy $\otimes$-equivalence relation eq. In case the size of the domain D is normalized to 1, equations (8) and (9) simplify to

$$eq(x, y) = 1 - \delta(x, y), \tag{14}$$

$$dis(y) = 1 - s(y). \tag{15}$$

3.6 Equality of extended lines has granularity

Two distinct exact coordinate points p and q determine a unique line, even if they are arbitrarily close to one another. For two distinct extended points P and Q this is not necessarily the case: if P and Q are "very close" to one another and the extended line L_1 is "too broad", then it may happen that P and Q behave like one single point with

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[2] Gerla uses the name *approximate similarity relation*. In the present abstract we use the name *approximate fuzzy equivalence relations* to stress the connection with equality relations. Since the terms fuzzy equivalence and similarity are used interchangeably in the fuzzy logic literature, this should not cause confusion.

respect to L_1 (Figure 3): Even though P and Q lie on L_1, they do not specify any directional constraint for L_1. Consequently, the graduated equality value $eq(L_1, L_2)$ may become arbitrarily small (i.e. zero).

As a consequence, an additional granularity factor gr must be added to $eq(L_1, L_2)$ for each of the involved extended lines L_1 and L_2. The granularity factor $gr[s(L_1), \delta(P,Q)]$ for L_1 depends on two parameters: the size $s(L_1)$ of L_1, and the distance $\delta(P,Q)$ of P and Q.
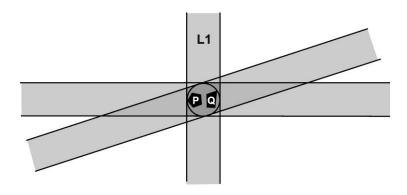

Figure 3. P and Q behave like one single point w.r.t. L_1.

4. Conclusions

We discussed Euclid's first postulate *"The line determined by two points is unique"* in the context of positional uncertainty of points and lines. With regard to the goal of establishing an analogous property for extended points and lines, we proposed a list of four requirements to an equality relation between extended *lines*: Such a relation should be 1. graduated, 2.weakly transitive, 3. inverse to distance, and 4. incorporate granularity. A full paper describing a formalization of Euclid's first postulate in fuzzy Łukasiewicz logic, which incorporates these requirements, is currently under preparation.

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