Buffer with Arcs on a Round Earth

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1. Introduction

The buffer of distance d about a geospatial object o is the set of all points whose distance from o does not exceed d. Buffer is one of the most fundamental constructions in geospatial computations. For example, it may be used to define the danger zone around the path of a hurricane, or as a query window for selecting all the objects in a database which are in the vicinity of a given object.

As long as the modeling of geospatial objects has been restricted to points and line segments, buffers could only be approximated by polygons. The introduction of circular arcs in commercial products such as [Oracle 2001] (echoed in drafts for future versions of international standards such as ISO 19125 and SQL/MM) affords exact representations for planar buffers.

The construction and processing of circular arcs relies on age-old geometric constructions in the plane, and these do not easily extend to curved surfaces. That is probably why the use of circular arcs in geospatial modeling has been restricted to planar maps (e.g. Section 5.2.3in [Oracle 2001]). In [Kallay 2010] we have recently proposed how circular arcs may be defined on ellipsoid earth models. Here we report how we have used this definition to model buffers of geospatial objects on an ellipsoid earth model.

2. Circular Arcs on an Ellipsoid Earth Model

On a sphere, any pair of distinct non-antipodal points defines a unique great-circle arc, which for the purpose of defining spherical polygons and line-strings is the "line segment" between them. In Microsoft's SQL Server™ SqlGeography class, the definition of line segments on an ellipsoid earth model relies on the *reference sphere* – a unit sphere on which every point represents the point on the ellipsoid with the same geodetic (longitude-latitude) coordinates. The "line segment" between a pair of (non-antipodal) points on the ellipsoid is defined as the great-circle arc between them on the ellipsoid's reference sphere ([Kallay 2007]).

The prevailing definition of a circular arc in the plane is in terms of 3 points – two endpoints and some additional arbitrary point on the arc. Similarly, three distinct points define a unique circular arc on a sphere. In [Kallay 2010] we have proposed a similar definition for circular arcs on an ellipsoid: *The circular arc defined by 3 points on the ellipsoid is the circular arc through their representatives on the reference sphere.* Since the mapping from the reference sphere to the ellipsoid introduces a slight distortion, such an arc may not be exactly circular on the ellipsoid, because it may not be equidistant from any center point. However, arcs that are circular on the ellipsoid can be approximated very efficiently with these reference sphere arcs. For

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example, on the WGS 84 ellipsoid ([Wikipedia 2010]), any circle can be approximated by 4 arcs with an error that does not exceed 0.06% of its radius.

3. Buffer Construction in the Plane

In the plane, the *offset point* $O(p,v,d)$ from a given point p at given distance d in the direction of a given vector v is the point in the direction of v at distance d from p.
The boundary of the buffer of distance d about a point p is a disk whose boundary can be defined with the two circular arcs $[O(p,u,d), O(p,v,d), O(p,-u,d)]$ and $[O(p,-u,d), O(p,-v,d), O(p,u,d)]$, where u and v are any pair of mutually perpendicular vectors.

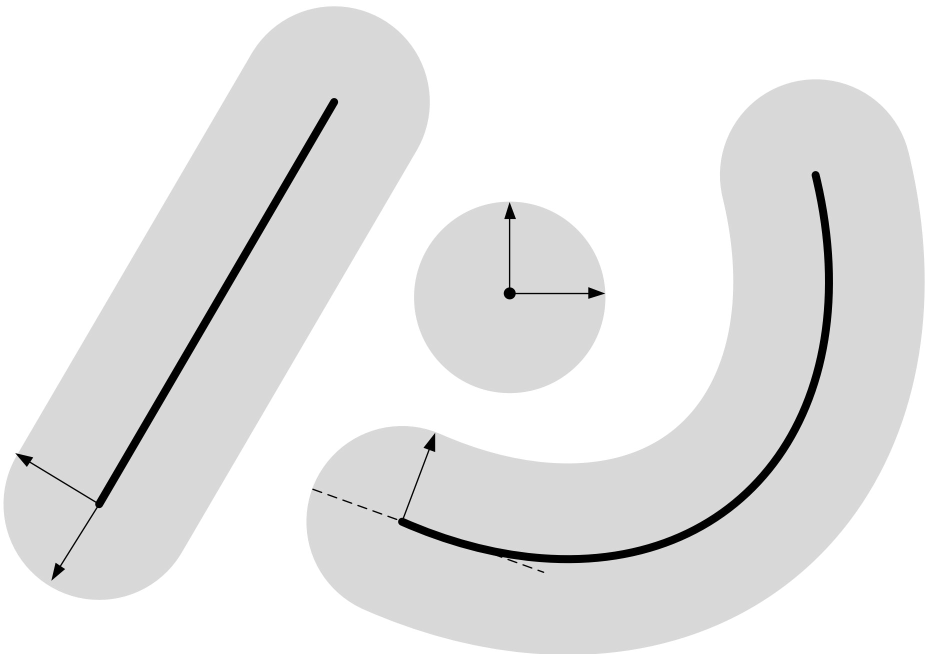

Figure 1: Planar buffers about a point, a line and an arc.

The boundary of the buffer about a line segment comprises the left and right *offset lines* connected by the round start and end *caps*. The offset line segments are $[O(s,v,d), O(e,v,d)]$ and $[O(e,-v,d), O(s,-v,d)]$, where s and e are the segment's start and end points respectively, and v is perpendicular to the direction of the vector $e-s$. The caps can be defined as the arcs $[O(s,-v,d), O(s,s-e,d), O(s,v,d)]$ and $[O(e,v,d), O(s,e-s,d), O(e,-v,d)]$.

Similarly, the boundary of the buffer about a circular arc comprises two *offset arcs* connected by round caps. In addition to the offset points of the arc's start and end points, the definition of a left or right offset uses the left or right offsets of the arc's midpoint. At any point, the offset direction is perpendicular to the arc's tangent vector. The midpoints of the start and end caps are offset along the start and end tangent vectors.

The buffer about a compound curve is constructed as the union of buffers about its line and arc segments. The buffer about a curved polygon (with linear and circular edges) is computed as the union of the polygon with the buffer of its boundary.
In an object model with no circular arcs, the offset lines of a line segment can be represented exactly, but the caps must be approximated by polygonal paths.

4. Buffer Construction on a Sphere

A construction of buffer on a sphere may follow the same pattern, with slight modifications:

- Distances between points are measured as arc length on the sphere.
- For the purpose of computing offset vectors, the direction of a segment at a point p is the segment's tangent vector at p. The offset directions are perpendicular to that in the sphere's tangent plane at p.
- The computation of union is performed in some planar projection, as outlined in [Kallay 2008].
- The offset of a line segment is in general not a line segment, so in the absence of arcs, edge offsets can only be approximated polygonally. The problem is illustrated in Figure 2.

As it does in the plane, the addition of circular arcs to a points-and-lines object model affords exact representation of end caps and corner rounding. But on a sphere, it also affords an exact representation for the offsets of line segments. And since the offsets of circular arcs are also circular arcs, an object model based on points, line segments and circular arcs is closed under the buffer operation. This is a very compelling reason for adding circular arcs to the object model on a spherical earth.

5. Buffer Construction on an Ellipsoid

The picture is somewhat more complicated on ellipsoid earth models. Our construction works as follows:

- We measure the distance between points as the length of the line segment between them, as defined in [Kallay 2007]. This is not exactly geodesic distance, but it is consistent with our STDistance method.
- We compute direction vectors in the tangent plane of the reference sphere, and offset directions are computed to be perpendicular to them in that plane. On the ellipsoid they are perpendicular only approximately, but the error introduced to the buffer computation by this compromise is negligible.
- Our proposed circular arcs are not exactly circular, and the offsets of line segments and circular arcs (under any definition) cannot be represented exactly as circular arcs (of any type). However, offset curves can be approximated very efficiently with circular arcs of the type defined in Section 2. We have used a simple subdivision algorithm for that:
  - A circular arc is initially constructed through the exact offsets of the start, end and mid points of the input line or arc.
  - The distance from the constructed arc to the input is then computed at two intermediate points, and compared to the prescribed distance.
  - When the deviation exceeds the prescribed tolerance, the arc is split at its midpoint, and the process is applied recursively to its two parts.

  For an error tolerance of $0.001d$, no arc ever needs to be subdivided more than twice.

Note that parallels (i.e. lines of constant latitude) are circles in the sense of Section 2, and by being equidistant from the poles they are exactly circular on the ellipsoid as well. It follows that buffers about the poles can be represented exactly by arcs of the type defined in Section 2. Since the distance between every pair of parallels is constant, the offset of parallels can also be represented exactly by Section 2 arcs.

6. Examples

For comparison, we have constructed a buffer of 3000 km about the line (-40 0, 40 0) on the equator in three separate modes. To illustrate the fact that the offset of a line segment is not a line segment, the first one is a polygonal approximation with a loose error tolerance (150 km). The deviation of the polygonal approximations (with two line segment each) from the theoretical offset curves is clearly visible in Figure 2.

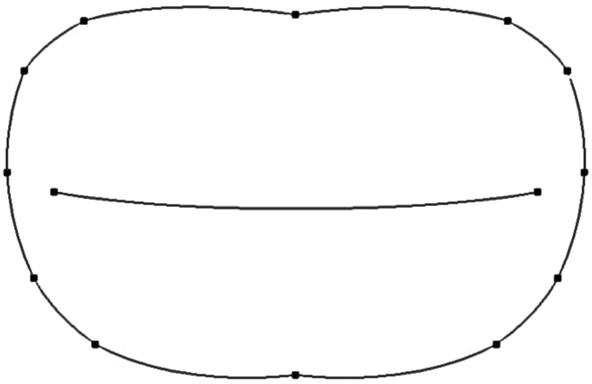

Figure 2: Polygonal approximation with loose error tolerance.

With a tight error tolerance (3 km), the polygonal approximation of the two offset curves in the second mode requires 28 linear edges. The approximation of the entire buffer requires 100 edges. The result is illustrated in Fig 3.

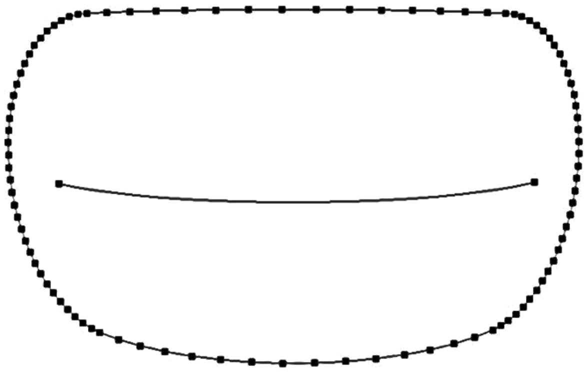

Figure 3: Polygonal approximation with tight error tolerance.

With circular arcs, the buffer is approximated with the same tight error tolerance (3 km) by 6 circular arcs. Since the input line lies on the equator, its offsets are segments of parallels, which are exactly Section 2 arcs. The caps cannot be represented exactly as arcs, and the approximation of each cap therefore requires two arcs. The result is illustrated in Figure 4.

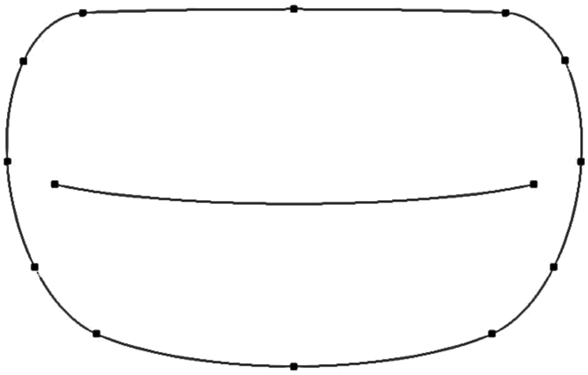

Figure 4: Circular approximation with tight error tolerance.

As a somewhat less contrived example, here is a 10 km buffer about the Mediterranean island of Cyprus, which is defined as a polygon with 134 vertices. With an error tolerance of 10 m, our polygonal approximation required 441 vertices. Using arcs, the same accuracy is achieved here with 253 vertices.

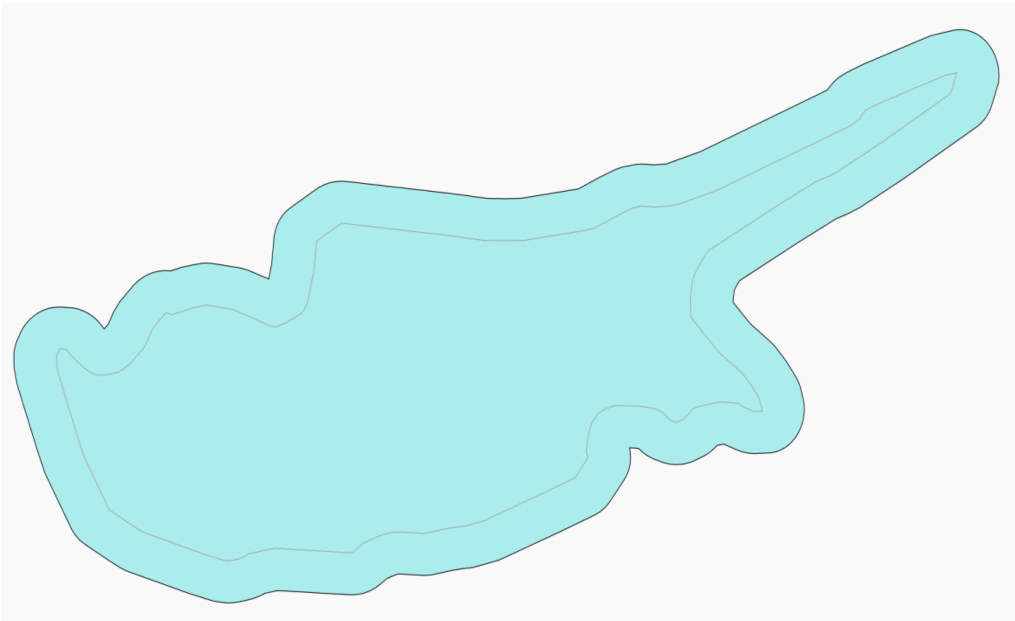

Figure 5: A 10 km buffer about Cyprus.

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